

Remarks on the harmonic oscillator with a minimal position uncertainty

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Abstract

We show that this problem gives rise to the same differential equation of a well known potential of ordinary quantum mechanics. However there is a subtle difference in the choice of the parameters of the hypergeometric function solving the differential equation which changes the physical discussion of the spectrum.

1 Introduction

The non-renormalizability of quantum gravity requires the introduction of an intelligent UV cutoff at the Planck scale, like non-commutative algebras. However this program is not easy to make explicit because many of them break basic principles of a quantum field theory like Lorentz invariance and unitarity.

It has been proposed that an effective cutoff in the ultraviolet should quantum theoretically be described as a non-zero minimal uncertainty Δx_0 in position measurements [1]-[2]. Technically it is necessary to require that there is no minimal uncertainty in momentum in order to use a continuous representation.

Recently it has also been found that this type of non-commutative cutoff is compatible with Lorentz invariance [3], in particular the Snyder geometry [4] (at the price of losing translational invariance).

Since there are experimental observations pointing out that Lorentz invariance is preserved even at the Planck scale [5], this theoretical framework is worth being investigated.

In general it has been studied how this intelligent cutoff affects the underlying quantum mechanical structure. The most interesting example is the harmonic oscillator with a minimal position uncertainty, which has been solved in [1].

In this letter we point out that their solution leads to the same differential equation of a well known potential of quantum mechanics. We show that the difference between the two problems is in a different sign of the parameters of the hypergeometric functions solving the differential equation leading to a different physical spectrum.

2 Minimal position uncertainty

While in ordinary quantum mechanics Δx can be made arbitrarily small, this is no longer the case if the following relation holds:

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + \gamma) \quad (2.1)$$

The following Heisenberg algebra generated by \hat{x} and \hat{p} obeying the commutation relations

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta \hat{p}^2) \quad (2.2)$$

underlies the uncertainty relation with $\gamma = \beta < p >^2$, from which

$$\Delta p = \frac{\Delta x}{\hbar\beta} \pm \sqrt{\left(\frac{\Delta x}{\hbar\beta}\right)^2 - \frac{1}{\beta} - \langle p \rangle^2} \quad (2.3)$$

we can read off the minimal position uncertainty

$$\Delta x_{min}(\langle p \rangle) = \hbar\sqrt{\beta}\sqrt{1 + \beta \langle p \rangle^2} \geq \hbar\sqrt{\beta} \quad (2.4)$$

In the algebra (2.2) there is no nonvanishing minimal uncertainty in momentum. The Heisenberg algebra (2.2) can be represented continuously on momentum space wave functions $\psi(p) = \langle p | \psi \rangle$ as

$$\begin{aligned} \hat{p}\psi(p) &= p\psi(p) \\ \hat{x}\psi(p) &= i\hbar(1 + \beta p^2)\frac{\partial}{\partial p}\psi(p) \end{aligned} \quad (2.5)$$

The position and momentum operators are symmetric on the domain S_∞ with respect to the following modified scalar product:

$$\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p)\phi(p) \quad (2.6)$$

The identity operator can thus be expanded as

$$1 = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} |p\rangle\langle p| \quad (2.7)$$

and the scalar product of momentum eigenstates is:

$$\langle p | p' \rangle = (1 + \beta p^2) \delta(p - p') \quad (2.8)$$

The position operator is no longer essentially self-adjoint but has a one parameter family of self-adjoint extensions. This means that even if formal eigenvectors for the position operator exist, they are not physical states since they have infinite energy.

3 The harmonic oscillator

From the expression for the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + m\omega^2 \frac{\hat{x}^2}{2} \quad (3.1)$$

and the representation for \hat{x} and \hat{p} in the p -space (2.5) we get the following form for the stationary state Schrodinger equation:

$$\frac{d^2\psi(p)}{dp^2} + \frac{2\beta p}{1 + \beta p^2} \frac{d\psi(p)}{dp} + \frac{4\beta(q - \beta r p^2)}{(1 + \beta p^2)^2} \psi(p) = 0 \quad (3.2)$$

where

$$q = \frac{E}{2m\hbar^2\omega^2\beta} \quad r = \frac{1}{4\beta^2 m^2 \hbar^2 \omega^2} \quad (3.3)$$

and E is the energy.

In order to find the explicit solution it is useful to introduce a new variable z in terms of which the poles coincide with those of the hypergeometric function, i.e. $0, 1, \infty$:

$$\frac{d^2\psi(z)}{dz^2} + \frac{2z - 1}{z(z - 1)} \frac{d\psi(z)}{dz} - \frac{(q + r(1 - 2z)^2)}{z^2(z - 1)^2} \psi(z) = 0 \quad z = \frac{1}{2} + i\frac{\sqrt{\beta}}{2}p \quad (3.4)$$

This type of differential equation is not new in physics since it is related to a well known potential of ordinary quantum mechanics:

$$V(x) = -\frac{U_0}{ch^2\alpha x} \quad (3.5)$$

The corresponding eigenvalue problem is, in the x -space representation, given by:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} \left(E + \frac{U_0}{ch^2\alpha x} \right) \psi(x) = 0 \quad (3.6)$$

The analysis of the spectrum is rather clear in this case. The positive energy spectrum is continuous and the negative energy spectrum is discrete. Let us remember the discussion in the last case. By changing variables $\xi = th(\alpha x)$ and introducing the notations:

$$\epsilon = \frac{\sqrt{-2mE}}{\hbar\alpha} \quad s(s + 1) = \frac{2mU_0}{\hbar^2\alpha^2} \quad (3.7)$$

we obtain the differential equation:

$$\frac{d}{d\xi} \left[(1 - \xi^2) \frac{d\psi}{d\xi} \right] + \left[s(s + 1) - \frac{\epsilon^2}{1 - \xi^2} \right] \psi = 0 \quad (3.8)$$

To reach the hypergeometric function it is necessary the substitution:

$$\begin{aligned}\psi &= (1 - \xi^2)^{\epsilon/2} w(\xi) \\ u &= \frac{1}{2}(1 - \xi)\end{aligned}\tag{3.9}$$

We end up with the hypergeometric equation:

$$u(1 - u)w'' + (\epsilon + 1)(1 - 2u)w' - (\epsilon - s)(\epsilon + s + 1)w = 0\tag{3.10}$$

The finite solution for $\xi = 1(x = +\infty)$ is

$$\psi = (1 - \xi^2)^{\epsilon/2} {}_2F_1 \left[\epsilon - s, \epsilon + s + 1, \epsilon + 1, \frac{1 - \xi}{2} \right]\tag{3.11}$$

The condition of finiteness of ψ at $\xi = -1(x = -\infty)$ requires $\epsilon - s = -n$ (where n is an integer), then the hypergeometric function reduces to a polynomial of degree n .

In this way, the energy levels are determined by the condition

$$s - \epsilon = n\tag{3.12}$$

from which

$$E_n = -\frac{\hbar^2 \alpha^2}{8m} \left[-(1 + 2n) + \sqrt{1 + \frac{8mU_0}{\alpha^2 \hbar^2}} \right]^2\tag{3.13}$$

The number of discrete energy levels is finite and determined by the condition $\epsilon > 0$ i.e.

$$n < s\tag{3.14}$$

The differential equation of the quantum mechanical problem (3.8) compared with the noncommutative problem (3.4) implies the following relation between the parameters

$$q = \frac{\epsilon^2 - s(s + 1)}{4} \quad r = \frac{s(s + 1)}{4} \quad \epsilon = 2\sqrt{q + r}\tag{3.15}$$

In the differential equation it appears only the combination ϵ^2 . We therefore want to clarify that while in the quantum mechanical problem the natural choice is the solution with

the positive sign $+\epsilon$, in the non-commutative problem the right choice turns out to be $-\epsilon$. In fact the non-commutative problem requires the substitution

$$\psi = (1 - \xi^2)^{-\epsilon/2} w(\xi) \quad u = \frac{1}{2}(1 - \xi) \quad (3.16)$$

which implies the following solution in the p variables ($\xi = -i\sqrt{\beta}p$):

$$\psi(p) = \frac{1}{(1 + \beta p^2)^{\sqrt{q+r}}} {}_2F_1 \left(-\epsilon - s, -\epsilon + s + 1, 1 - \epsilon, \frac{1}{2} + i\frac{\sqrt{\beta}}{2}p \right) \quad (3.17)$$

The quantization condition is now

$$\epsilon + s = n \quad (3.18)$$

and for q we obtain

$$q = \frac{(n - s)^2}{4} - r \quad (3.19)$$

The requirement $\epsilon > 0$ i.e. $n - s > 0$ is always satisfied for every integer n by choosing also for s the negative solution

$$s = -\frac{1}{2} - \frac{\sqrt{1 + 16r}}{2} \quad (3.20)$$

The number of energy levels is now infinite and not subjected to any condition

$$\begin{aligned} E_{level} &\sim \hbar\omega \frac{q}{\sqrt{r}} = \hbar\omega \left(\frac{(n - s)^2}{4\sqrt{r}} - \sqrt{r} \right) = \\ &= \hbar\omega \left[\left(n + \frac{1}{2} \right) \left(\sqrt{1 + \frac{1}{16r}} + \frac{1}{4\sqrt{r}} \right) + \frac{n^2}{4\sqrt{r}} \right] \end{aligned} \quad (3.21)$$

For $\beta \rightarrow 0$ ($r \rightarrow \infty$) we recover the harmonic oscillator energy levels.

The conclusion is that it is enough to choose a different sign ($\pm\epsilon$) in the solution of the same differential equation (3.4) to reach completely different physical spectra.

References

- [1] A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. **D 52** (1995) 1108, hep-th/9412167.

- [2] A. Kempf, G. Mangano, Phys. Rev **D 55** (1997) 7909, hep-th/9612084.
- [3] A. Bose, arXiv:1011.5234v1[physics.gen-ph].
- [4] H. S. Snyder, Phys. Rev **71** (1947) 38.
- [5] V. A. Kostelecky, N. Russell, Rev. Mod. Phys. **83** (2011) 11, arXiv:0801.0287v3[hep-ph].